GEOMETRIC GROUP THEORY - BACKPAPER EXAM.

Time : 3 hours Max. marks : 100

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a proof.

(1) Let G, H be the groups defined by the presentations

$$G \left\langle a, b \,/\, abab^{-1} \right\rangle, \quad H = \left\langle u, v \,/\, u^2 v^2 \right\rangle.$$

(a) Prove that $G \cong H$.
(b) Show that G is an infinite group and the elements a, b are of infinite order.
(c) Show that G is non-abelian.
(d) Show that the center $Z(G) \neq \{1\}$.
(e) Show that the abelianization G_{ab} of G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.
(f) Exhibit H as a free product with amalgamation.
(g) Exhibit G as an HNN-extension.
(h) Is G virtually cyclic?
(i) Show that G does not contain a non-abelian free group. [$9 \times 4 = 36$]

(2) (a) Let the groups G, H have presentations

$$G = \langle X/R \rangle, \quad H = \langle Y/S \rangle.$$

Show that

$$\left\langle X, Y/R, S, xyx^{-1}y^{-1}, x \in X, y \in Y \right\rangle$$

is a presentation of the group $G \times H$. [12]

(b) When are two metric spaces quasi-isometric? Show that a homomorphism $f : G \longrightarrow H$ between two finitely groups is a quasi-isometry if and only if both $\ker(f)$ and $\mathrm{coker}(f)$ are finite. [2+10]

(3) (a) Let G be a finitely generated group with a finite generating set X. Discuss the definition of the growth function of G with respect to X. Compute the growth functions of the groups $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ with respect to finite generating sets of your choice. Using this, or otherwise, show that $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are not quasi-isometric. [4+10+6]

(b) Define three (equivalent) notions of hyperbolicity of a geodesic metric space. Show that there does not exist a quasi-isometric embedding $f : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$. Is there a quasi-isometric embedding $g : \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$? [6+8+6]